Cooperative Network Localizability via Semidefinite Programming

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***Abstract*—In cooperative localization, the aim is to compute the locations in Euclidean space of a set of nodes performing pair-wise distance measurements. In cases of lack of measurements, several nodes might have multiple feasible solutions meeting the distance constraints. In this paper, we are interested in identifying the nodes that have a unique solution.**

By employing a semidefinite programming (SDP) formulation of the problem, it is possible to identify only a portion of the uniquely solvable nodes. To improve the identification of these nodes, we develop an iterative algorithm based on SDP. At each iteration, the objective function of the SDP problem is modified in order to identify additional uniquely solvable nodes.

We apply this algorithm to study the statistical occurrence of uniquely solvable nodes in uniformly generated networks, and compare the results with the simple SDP. We also investigate the errors in the computed locations for both methods and a variant of the SDP method augmented by bounding constraints on unobserved distances.

I. Introduction

Finding the locations of a set of wireless communicating devices has a lot of practical applications, going from the deployment of ad-hoc networks and its related topics, e.g. communication enhancement and location-based routing, toward a variety of location-based services and applications, e.g. military, environmental and health.

When pairs of devices (nodes) perform measurements relevant to their relative locations, they can be localized in a Cartesian coordinate system. Several names have been attributed to this topic in the literature, such as network calibration, cooperative and self-localization [1].

In general, not all pairs of nodes perform measurements and several nodes might have ambiguities on their location solutions. The detection of the possible ambiguities is of great importance as it allows to deal with them: either by avoiding them and improving the robustness of the solutions [2], or by mitigating them by making additional measurements or using more a priori information [3].

In this work, we consider error free distance measurements, and we are interested in identifying the nodes that have a unique solution verifying the distance constraints. In real applications, measurements are affected by errors, but considering error free ones can have several justifications: e.g., the nodes locations are already estimated by an appropriate estimation method [4] and one wants to test the uniqueness of the estimates, or the true locations are known and one is studying the statistical occurrence of the uniquely solvable nodes.

One approach to study the uniqueness is graph rigidity theory [5]. A graph can be associated to a network by associating a vertex to each node and connecting two vertices if their separating distance is known. For a network lying in a 2-dimensional space and having the nodes in generic positions (i.e., no collinear or collocated nodes), it is shown in [6] that a sufficient (but not necessary) condition for a node to have a unique solution is the following:

- the vertex corresponding to this node in the network graph G belongs to a globally rigid subgraph of G,
- and the vertices corresponding to at least three anchor nodes of known locations belong to this subgraph.

This test is not applicable when nodes are not in generic positions (e.g., a planned deployment where several nodes are collinear).

Another approach is to use SDP. It is shown in [7] that by formulating the network localization problem as an SDP problem with an appropriate relaxation of the constraints, all the nodes of any uniquely localizable network can be identified and correctly localized. When the network is not uniquely localizable, some nodes having a unique solution are wrongly localized by the SDP method which does not allow to decide about the uniqueness of their solutions. In [3], the SDP solution is post-processed by a steepest descent algorithm, and in [8], the objective function of the SDP formulation is modified to maximize the sum of lengths of some unobserved edges. While these two solutions can correctly localize additional nodes, they do not tell us whether these nodes have a unique solution.

In this paper, we develop an algorithm based on SDP method that improves the identification of the uniquely solvable nodes. It performs by modifying the objective function of the SDP problem in order to minimize the rank of the optimal solution. This algorithm is implemented iteratively where we modify the objective function at each iteration and try to identify additional nodes.

The paper is organized as follows. In section II, the network localization problem is presented and the SDP method is formulated and applied to localizability testing. In section III, the iterative SDP-based algorithm is developed. And in section IV, it is compared statistically to the simple SDP method in terms of the detection of uniquely solvable nodes. This comparison is done via Monte Carlo simulations. In section V, the accuracy of the computed locations is investigated and compared to another formulation of the SDP method incremented by bounding constraints on unobserved distances. Concluding remarks are presented in section VI.

II. SDP Method and Localizability Test

In this section, we present the network localization problem, define the term of unique localizability, and show how the SDP method identifies the uniquely localizable nodes.

We use the following notations. $\mathbf{I}_2$ is the identity matrix of rank 2. $\mathbf{0}$ is the column vector of zeros. $\mathbf{e}_i$ is the column vector with 1 at the i^{th} position and zeros elsewhere. $\mathbf{e}_{ij} = \mathbf{e}_i - \mathbf{e}_j$. The dimensions of the vectors will be clear in the context. $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is the Euclidean norm, and $\langle , \rangle$ is the inner product. $(\mathbf{u}; \mathbf{v}) = [\mathbf{u}^T \mathbf{v}^T]^T$ and $\mathbf{v}^T$ denotes the transpose of $\mathbf{v}$. $\mathbb{R}$ is the field of real numbers.

A. Network Localization Problem

Throughout this paper, we consider networks lying in a 2-dimensional space ($2D$). The results can be extended to the 3-dimensional case straightforwardly.

A network of size N consists of $m < N$ anchor nodes of known locations $\mathbf{a}_k \in \mathbb{R}^2$, $k = 1, \ldots, m$, and $n = N - m$ target nodes of unknown locations $\mathbf{x}_j \in \mathbb{R}^2$, $j = 1, \ldots, n$. $\mathbf{a}_k$ and $\mathbf{x}_j$ are column vectors. Nodes are denoted by their location vectors, i.e., node $\mathbf{x}_j$ denotes the j^{th} target node out of n.

The distance between an anchor node $\mathbf{a}_k$ and a target node $\mathbf{x}_j$ is denoted by $\bar{d}_{kj} = \|\mathbf{a}_k - \mathbf{x}_j\|$. And the distance between two target nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ is denoted by $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$.

The distance between two nodes is known (and the nodes are said to be connected) if and only if (iff) this distance is less than a connectivity range R.

Let $\mathcal{N}_A = \left\{(k, j) : \bar{d}_{kj} \text{ is known}\right\}$ and $\mathcal{N}_T = \left\{(i, j) : d_{ij} \text{ is known}\right\}$.

The network localization problem can be stated as follows:

$$\begin{array}{lll} \text{find} & \mathbf{x}_j, j = 1, \ldots, n, & \\ \text{s.t.} & \|\mathbf{a}_k - \mathbf{x}_j\|^2 = \bar{d}_{kj}^2 & \forall (k, j) \in \mathcal{N}_A \\ & \|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2 & \forall (i, j) \in \mathcal{N}_T. \end{array} \quad (1)$$

Let $\bar{\mathbf{X}} = [\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_n] \in \mathbb{R}^{h \times n}$, $h \geq 2$. We admit that $\bar{\mathbf{X}}$ is an h-dimensional solution of the network localization problem iff it verifies the following equations:

$$\begin{array}{ll} \|(\mathbf{a}_k; \mathbf{0}) - \bar{\mathbf{x}}_j\|^2 = \bar{d}_{kj}^2 & \forall (k, j) \in \mathcal{N}_A \\ \|\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_j\|^2 = d_{ij}^2 & \forall (i, j) \in \mathcal{N}_T, \end{array} \quad (2)$$

where $(\mathbf{a}_k; \mathbf{0}) = [\mathbf{a}_k^T \ \mathbf{0}^T]^T$ is an h-dimensional vector.

Now, we define the term of unique localizability:

Definition 1: A target node $\mathbf{x}_j$ is $2D$ uniquely localizable iff for every pair of 2-dimensional solutions $\bar{\mathbf{X}}$ and $\bar{\mathbf{X}}'$, the equation $\left(\bar{\mathbf{X}} - \bar{\mathbf{X}}'\right) \mathbf{e}_j = \mathbf{0}$ is verified, or in other words, the node has a unique solution verifying the constraints in (1).

Definition 2: A target node $\mathbf{x}_j$ is uniquely localizable iff for every $h > 2$ and for every pair of h-dimensional solutions $\bar{\mathbf{X}}$ and $\bar{\mathbf{X}}'$, the equation $\left(\bar{\mathbf{X}} - \bar{\mathbf{X}}'\right) \mathbf{e}_j = \mathbf{0}$ is verified.

Unique localizability implies $2D$ unique localizability as the network has at least one 2-dimensional solution.

Definition 3: The network is $2D$ uniquely localizable iff all its target nodes are $2D$ uniquely localizable.

Definition 4: The network is uniquely localizable iff all its target nodes are uniquely localizable.

B. SDP Formulation

The SDP formulation of the network localization problem is derived in [9]. We will present it here for self completeness of the paper.

Problem (1) can be written in matrix form as follows:

$$\begin{array}{lll} \text{find} & \mathbf{X} \in \mathbb{R}^{2 \times n}, \mathbf{Y} \in \mathbb{R}^{n \times n}, & \\ \text{s.t.} & \mathbf{e}_{ij}^T \mathbf{Y} \mathbf{e}_{ij} = d_{ij}^2 & \forall (i, j) \in \mathcal{N}_T \\ & (\mathbf{a}_k; -\mathbf{e}_j)^T \mathbf{Z} (\mathbf{a}_k; -\mathbf{e}_j) = \bar{d}_{kj}^2 & \forall (k, j) \in \mathcal{N}_A \\ & \mathbf{Y} = \mathbf{X}^T \mathbf{X}, & \end{array} \quad (3)$$

where $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ and $\mathbf{Z}$ is defined as follows:

$$\mathbf{Z} = \left[\begin{array}{cc} \mathbf{I}_2 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{array} \right]. \quad (4)$$

Problem (3) is a nonconvex optimization problem. It can be transformed into a semidefinite program by relaxing the equality constraint $\mathbf{Y} = \mathbf{X}^T \mathbf{X}$ into a semidefinite condition $\mathbf{Y} - \mathbf{X}^T \mathbf{X} \succeq 0$, which is equivalent to $\mathbf{Z} \succeq 0$. Then, we arrive at the following SDP problem:

$$\begin{array}{lll} \text{minimize} & 0, & \\ \text{s.t.} & \mathbf{Z}_{1:2,1:2} = \mathbf{I}_2 & \\ & (\mathbf{0}; \mathbf{e}_{ij})^T \mathbf{Z} (\mathbf{0}; \mathbf{e}_{ij}) = d_{ij}^2 & \forall (i, j) \in \mathcal{N}_T \\ & (\mathbf{a}_k; -\mathbf{e}_j)^T \mathbf{Z} (\mathbf{a}_k; -\mathbf{e}_j) = \bar{d}_{kj}^2 & \forall (k, j) \in \mathcal{N}_A \\ & \mathbf{Z} \succeq 0. & \end{array} \quad (5)$$

C. Localizability Test

Let $\bar{\mathbf{Z}}$ be a solution of problem (5), then $2 \leq \text{rank}\bar{\mathbf{Z}} \leq 2 + n$. A max-rank solution is a solution that has the highest rank among all feasible ones. Such a solution can be computed by means of an interior-point algorithm.

Theorem 1 ([7]): Let $\bar{\mathbf{Z}}$ be a max-rank solution of problem (5), then the following statements are equivalent:

- The network is uniquely localizable.
- The rank of $\bar{\mathbf{Z}}$ is equal to 2.
- $\bar{\mathbf{Z}}$, represented as (4), satisfies $\bar{\mathbf{Y}} = \bar{\mathbf{X}}^T \bar{\mathbf{X}}$.

Thus, by finding a max-rank solution of (5), we can answer the question whether the network is uniquely localizable. We can also test the unique localizability of the different nodes according to the following important properties [7]:

- Node $\mathbf{x}_j$ is uniquely localizable iff $\bar{\mathbf{Y}}_{jj} - \|\bar{\mathbf{x}}_j\|^2 = 0$, where $\bar{\mathbf{Y}}_{jj}$ is the (j, j) entry of $\bar{\mathbf{Y}}$ and $\bar{\mathbf{x}}_j$ is the j^{th} column of $\bar{\mathbf{X}}$.
- Node $\mathbf{x}_j$ is not uniquely localizable iff $\bar{\mathbf{Y}}_{jj} - \|\bar{\mathbf{x}}_j\|^2 > 0$.

We shall call this test the simple SDP-based test in the sequel.

According to definitions 1 and 2, the uniquely localizable nodes constitute only a subset of the $2D$ uniquely localizable nodes, and the simple SDP-based test cannot detect all these latters. Fig. 2 provides an example.

III. Improving The Localizability Test

In this section, we develop a new solution for improving the identification of the $2D$ uniquely localizable nodes. We shall call it the iterative SDP-based algorithm (I-SDP).

We begin by showing how to reduce the rank of the SDP solution for a network of 3 nodes by a simple modification of the objective function of the SDP problem (5).

A. Rank Reduction via Scalar Product

We consider the network of Fig.1 consisting of two anchor nodes $\mathbf{a}_1$ and $\mathbf{a}_2$ and one target node $\mathbf{x}_1$.

By solving the SDP problem (5) corresponding to this network, we obtain a matrix $\bar{\mathbf{Z}}$ of rank 3, as node $\mathbf{x}_1$ is not uniquely localizable :

$$\bar{\mathbf{Z}} = \begin{bmatrix} \mathbf{I}_2 & \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_1^T & \bar{\mathbf{Y}} \end{bmatrix}, \quad (6)$$

where $\bar{\mathbf{Y}} > \|\bar{\mathbf{x}}_1\|^2$.

This result can be seen as if the SDP solver finds a 3-dimensional vector $\left(\bar{\mathbf{x}}_1; \pm\sqrt{\bar{\mathbf{Y}} - \|\bar{\mathbf{x}}_1\|^2}\right)$ and the computed location $\bar{\mathbf{x}}_1$ is the orthogonal projection of this vector on the plane of the network.

Let $\mathbf{a}_{1,2} = \mathbf{a}_2 - \mathbf{a}_1$. The constraint of problem (5)

$$(\mathbf{a}_k; -1)^T \mathbf{Z} (\mathbf{a}_k; -1) = \bar{d}_{k1}^2 \quad k = 1, 2 \quad (7)$$

is verified by $\bar{\mathbf{Z}}$, which implies that $\langle \mathbf{a}_{1,2}, \bar{\mathbf{x}}_1 \rangle = \langle \mathbf{a}_{1,2}, \mathbf{x}_1 \rangle =$ constant, or in other words, the computed location lies on the line perpendicular to the segment joining $\mathbf{a}_1$ and $\mathbf{a}_2$ and passing by the true position $\mathbf{x}_1$.

Let $\mathbf{a}_{1,2}^{\perp}$ be a vector perpendicular to the vector $\mathbf{a}_{1,2}$. The scalar product $\left\langle \mathbf{a}_{1,2}^{\perp}, \bar{\mathbf{x}}_1 - \mathbf{a}_1 \right\rangle$ is maximized when $\bar{\mathbf{Y}} = \|\bar{\mathbf{x}}_1\|^2$. Thus, we can obtain a solution in the plane of the network and reduce the rank of $\bar{\mathbf{Z}}$ to 2 by modifying the objective function of (5) to one of the following two linear functions:

$$\text{minimize} \quad \pm \left\langle \mathbf{a}_{1,2}^{\perp}, \mathbf{X} \right\rangle. \quad (8)$$

Node $\mathbf{x}_1$ is not $2D$ uniquely localizable. The two solutions obtained for each of the objective functions (8) are plotted in Fig. 1. These solutions, denoted by $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_1'$, verify $\bar{\mathbf{Y}} - \|\bar{\mathbf{x}}_1\|^2 = 0$ and $\bar{\mathbf{Y}}' - \|\bar{\mathbf{x}}_1'\|^2 = 0$.

B. Iterative SDP-Based Algorithm

We consider a network deployed in a plane with given anchors locations and distance measurements.

In the first step of the algorithm, the SDP problem (5) is solved and the uniquely localizable target nodes, identified by the simple SDP-based test described in II-C, are promoted to anchor nodes. Then, for a target node $\mathbf{x}_j$ connected to two anchor nodes $\mathbf{a}_k$ and $\mathbf{a}_l$, the following steps are processed:

- The two SDP problems with the following two objective functions (9) are solved:

$$\text{minimize} \quad \pm \left\langle \mathbf{a}_{k,l}^{\perp}, \mathbf{X}\mathbf{e}_j \right\rangle. \quad (9)$$

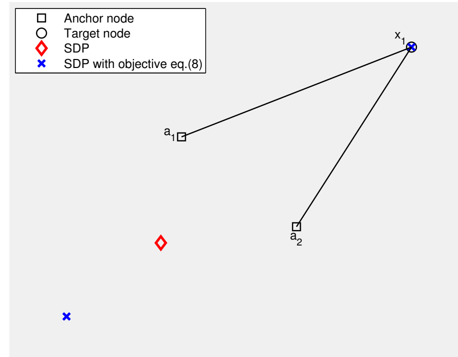

Fig. 1. A network of one target node connected to two anchor nodes.

  The obtained solutions are denote by $\bar{\mathbf{Z}}$ and $\bar{\mathbf{Z}}'$.
- If $\bar{\mathbf{Y}}_{jj} - \|\bar{\mathbf{x}}_j\|^2 = 0$ and $\bar{\mathbf{Y}}_{jj}' - \left\|\bar{\mathbf{x}}_j'\right\|^2 > 0$ (or vice versa), then node $\mathbf{x}_j$ and all the nodes $\mathbf{x}_i$ verifying $\bar{\mathbf{Y}}_{ii} - \|\bar{\mathbf{x}}_i\|^2 = 0$ are $2D$ uniquely localizable. These nodes are promoted to anchor nodes.
- Otherwise the node is omitted but it can be revisited later.

The algorithm performs iteratively by testing all the targets connected to two anchors until no more targets can be promoted. The different steps are summarized in Table I.

TABLE I
Iterative SDP-based Algorithm

0:	Solve the SDP problem (5)
1:	Promote uniquely localizable targets to anchors
2:	Find the set S of targets connected to two anchors
3:	For each target $\in S$
4:	Solve the 2 SDP problems with objective functions (9)
5:	If the selected target is $2D$ uniquely localizable
6:	Promote the identified targets to anchors
7:	Go to 2
8:	End If
9:	End For

C. Examples

Here, we provide two examples to show the efficiency of the I-SDP algorithm in situations where the graph rigidity test and the simple SDP-based test fail in identifying uniquely solvable nodes.

In Fig. 2, the two target nodes are non-uniquely localizable although the network is globally rigid. By applying the I-SDP algorithm, we can show that they are $2D$ uniquely localizable.

In Fig. 3, node $\mathbf{x}_1$ is not uniquely localizable and does not belong to a globally rigid subgraph, while it is $2D$ uniquely localizable and can be identified and correctly localized by the I-SDP algorithm.

IV. Localizable Nodes Occurrence

To illustrate the performance of the I-SDP algorithm, Monte Carlo simulations were performed for different network sizes N by taking $m = 3$ anchors and the connectivity range $R = 0.4$. 1000 networks are uniformly drawn for each value of N.

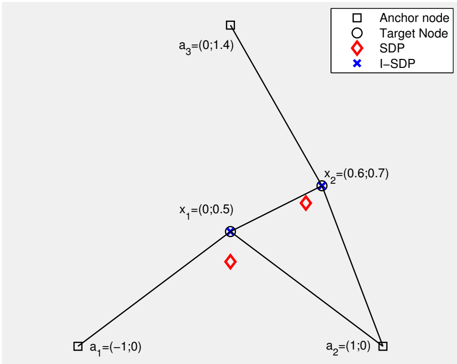

Fig. 2. A globally rigid network and $2D$ uniquely localizable but not uniquely localizable. Node $\mathbf{x}_2$ is not aligned with nodes $\mathbf{a}_2$ and $\mathbf{a}_3$ and the network can have a 3-dimensional solution verifying (2).

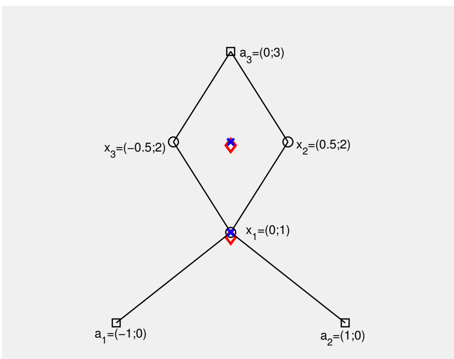

Fig. 3. Node $\mathbf{x}_1$ does not belong to a globally rigid subgraph and is not uniquely localizable but it is $2D$ uniquely localizable. A 3-dimensional solution can be obtained by rotating $\mathbf{x}_1$ around the axis formed by $\mathbf{a}_1$ and $\mathbf{a}_2$.

The nodes are uniformly distributed inside a square area of 1×1.

Fig. 4 presents the probability distribution of the number of nodes that are identified as uniquely solvable ($2D$ uniquely localizable) using the I-SDP algorithm, and the probability distribution of the number of uniquely localizable nodes identified by the simple SDP-based test. The two tests have been applied to the same generated networks.

We can see that when applying the I-SDP, the probability that all the nodes are uniquely solvable is increased and more uniquely solvable nodes are detected

We can also notice that the occurrence of the nodes that are $2D$ uniquely localizable but not uniquely localizable is small. For $N = 20$ nodes, about 20% of the networks are found to be $2D$ uniquely localizable (all the nodes have a unique solution), and about 17.5% are uniquely localizable, thus, $2D$ uniquely localizable networks are uniquely localizable in at most 88% of the time for this scenario.

V. Localization Accuracy

In this section, the accuracy of the locations computed by the simple SDP and the I-SDP is investigated in terms of mean location error.

The simulation scenarios are similar to those of the previous section except that the considered networks are connected. The nodes that are correctly localized by the I-SDP but not by the simple SDP are called ‘uncommon nodes’, these nodes are $2D$ uniquely localizable but not uniquely localizable. And the nodes that are not correctly localized by both methods are called ‘common nodes’.

Fig. 5 depicts the mean location error as a function of the network size N. We can see that the mean location error of uncommon nodes is smaller than that of common nodes. The uncommon nodes are $2D$ uniquely localizable and are connected to more nodes than the common nodes. As this error is small, we can deduce that SDP based localization methods provide a good starting point to apply a descent optimization solution.

We can also remark that the mean location error does not decrease with N but we mention that the occurrence of nodes having errors on their locations decreases with N as can be deduced from Fig. 4.

We also applied the variant of the SDP formulation with edge-bounding described in [3], where additional constraints are introduced on unmeasured distances in order to mitigate the flip ambiguities. The results are also plotted in Fig. 5 where we can notice that the error is decreasing with N for the common nodes as the number of constraints is increasing with N.

VI. Conclusion

In this paper, we developed an iterative algorithm based on SDP for improving the identification of the $2D$ uniquely localizable nodes. We used this algorithm and the simple SDP to study the statistical occurrences of the different kinds of nodes for specific scenarios. The computation of statistical occurrences can be also useful when deciding on the needed number of anchor nodes or communication ranges that guarantee a high probability of unique solvability. We also investigated the mean location errors and showed that $2D$ uniquely localizable nodes have small errors when they are localized by the simple SDP although they are not detected as uniquely solvable.

This work can be extended to 3-dimensional networks by selecting a node connected to 3 non-collinear anchor nodes at each iteration of the I-SDP.

It would be interesting to use the SDP based algorithms to study the robustness of the location estimates when distance measurements are erroneous.

VII. Acknowledgement

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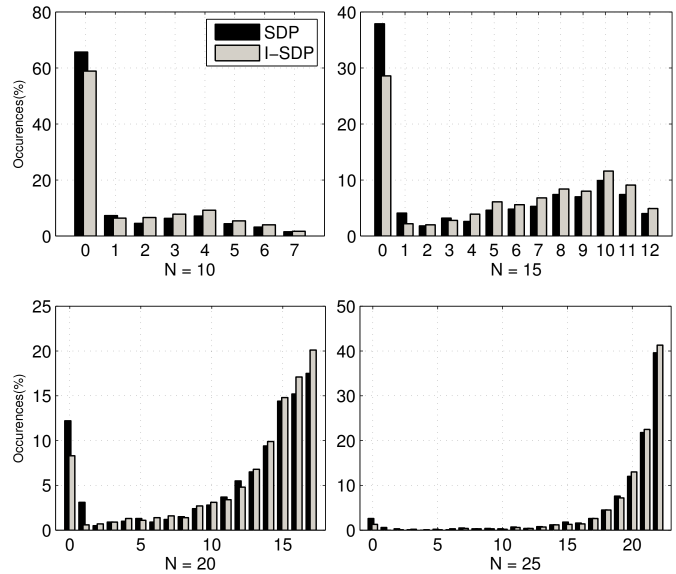

Fig. 4. Distribution of the number of uniquely solvable target nodes detected using the simple SDP and the I-SDP tests for different network sizes N. The number of anchor nodes is 3.

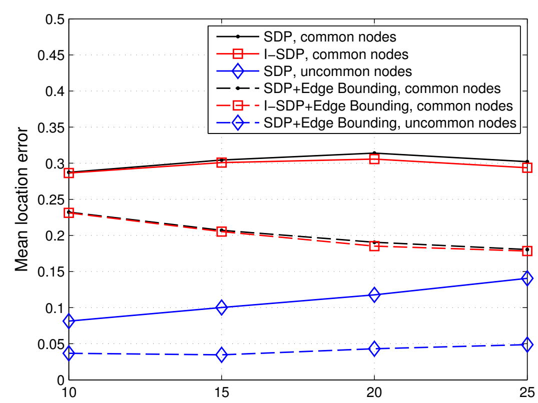

Fig. 5. Mean location error as a function of network size N.